Nature-of-Science Teaching: notes on the Lagrangian Methods in Maxwell's Electromagnetic Theory

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Abstract: About Maxwell's electromagnetic theory, some research and Nature-of-Science foundational questions arise: what are physics and mathematical objects in a theory and their shared objectivity and knowledge? What kind of mathematics did he prefer to use for his physics-mathematics electromagnetic theory? In this paper, based on previous publications by one of us (RP), we shortly present some results on Maxwell's electromagnetic theory, taking into account his Lagrangian methods and its specific formulation.

Keywords: Electromagnetic field, Maxwell, Lagrangian method.

1. Introduction

In both *A treatise on electricity and magnetism* (1873, 2 vols.) and *A dynamical theory of the electromagnetic field* (1864), James Clerk Maxwell tried to put into a mathematical language the *Experimental Researchers in Electricity* by Michael Faraday (1839-1855). For this purpose Maxwell went beyond the Newtonian approach reaching a new physics mathematics based on the concept of energy instead of that of force. A question arises: what is the original concept of energy that Maxwell had to deal with? Moreover: what kind of mathematics did he use for his physics-mathematics of the electromagnetic theory? In his investigation Maxwell used Lagrange's mathematics as a new approach to physical theory through the idea of connected mechanical system described by means of Lagrange's equations.

Maxwell laid the foundation for a new theory of electricity and magnetism as a new physical theory to be included in the physics-mathematics context. Therefore, he required a new dynamical theory for the description of this phenomenon and he built something unlike Newtonian mathematics; that is, Newton through Lagrange's equations (Maxwell 1873, III, part IV, chapter V, pp. 183-194).

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2. On Maxwell's Concept of Energy

In *On the induction of a current on itself* (Maxwell 1873, III, part IV, chapter IV, pp. 180-183), Maxwell explored some phenomena like the inductive action exerted by a current on neighbouring conductors and on the same conductor that carries the current. In his investigation he tried to make an analogy between only one wire conveying a current and a fluid flowing in a continued stream by means of the concepts of inertia and momentum. The aim was to use this comparison to build a new dynamical theory of electromagnetic phenomena.

Analysing the motion of a fluid in a pipe, it is observed that the effects of inertia of the fluid in the tube depend only on the internal characteristics of the system and they do not depend on anything outside the tube. Otherwise the same wire through by the same current exhibits a different behaviour because it differs if its form changes or if other bodies are present even when the wire keeps its form unchanged.

> It is difficult, however, for the mind which has once recognised the analogy between the phenomena of self-induction and those of the motion of material bodies, to abandon altogether the help of this analogy, or to admit that it is entirely superficial and misleading (Maxwell 1873, III, part IV, chapter IV, p. 181).

Maxwell also observed that if the phenomena of self-induction are due to momentum, this momentum has not to be that of electricity in the wire. Another question arises. When a current is set up in a circuit, the circuit itself acquires the ability to work, that is some kind of energy, maybe a sort of kinetic energy, "the energy which a moving body has in virtue of its motion" (Maxwell 1873, III, part IV, chapter IV, p. 182). Despite this hypothesis may seem reasonable, Maxwell emphasized that electricity in a wire can not be regarded as a moving body because of its dependence on external environment. Moreover another question could be stressed: are there some motion going on in the space outside the wire conveying a current?

In closing chapter four, Maxwell points out that in the following chapters it will analyse the consequences of the hypothesis that electrical phenomena can be studied by analogy with bodies in motion rather than investigate the reasons for such assumptions. In the following chapters, in fact, the author will try to deduce a theory of electricity from a dynamic point of view, assuming that the motion is communicated from one part of system to another by forces. The nature of the forces and the laws they obey are not explored because these forces will be eliminated from the equation of motion by means of the Lagrange's method.

After this careful consideration, the author focuses on the Lagrangian formalism need to build a new dynamic theory of electromagnetism rather than further explore the reasons for his conclusions.

3. Lagrangian Method in Maxwell's *Treatise*

A *dynamical Lagrangian system* is a dynamical system whose equations of motion are established by means of the principle of least action. The Lagrangian[2] *L*, which is the mathematical function that describes the dynamics of the system, in classical mechanics is generally defined by the difference between the kinetic energy and the potential energy of the system.

To build a dynamic theory of electromagnetism, Maxwell needed a mathematical approach suitable to be applied to different physical contexts. Since the introduction to chapter V, the author underlines that the aim of the new electric theory must be a close relationship between physics and its mathematical description. Maxwell then used the Lagrangian methods and related mechanics instead of the Newtonian one. The latter was not adapt to the nearest physical analysis of phenomena made by Faraday. Moreover, Maxwell (1873, III, part IV, chapter V, p. 184) recalls that, in his formulation of mechanics, Lagrange submits the dynamics to the power of mathematics, describing the dynamic relationship between the parts of a system by means of purely algebraic relations.

From a mathematical point of view, Lagrange's method consists in eliminating those quantities from the final equations that describe the interactions between the parts of a system permitted by physical constraints.

> In following the steps of this elimination the mind is exercised in calculation, and should therefore be kept free from the intrusion of dynamical ideas (Maxwell 1873, III, part IV, chapter V, p. 184).

Alongside these considerations, however, Maxwell stresses the importance of a continuous reference to dynamic, so that from the pure mathematical language can emerge the properties of moving bodies.

In *On the equations of motion of a connected system* (Maxwell 1873, III, part IV, chapter V, pp. 184-185) and throughout the rest of the fifth chapter of the *Treatise*, Maxwell considered the Lagrangian description of a system as a connected one, describing motion and energy relationship within the system as a whole, rather than in terms of laws concerning the actions of forces (Pisano 2013a, b). In other words, he has described the system as a whole rather than to consider it consists of parts as in the Newtonian approach.

After describing the characteristics of a connected system, Maxwell relates the possibility to express the kinetic energy *T* of the system in terms of variables and momenta, or in terms of momenta and velocities (Maxwell 1873, III, part IV, chapter V, pp. 184-186).

Then Maxwell's attention is focused on a third description of kinetic energy in terms of velocities and of variables, rather than on momenta and velocities.

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[2] See (Lagrange 1788); see also (Euler [1750], 1752, 1736, 1775). On mechanics, see recently (Pisano, Agassi, Drozdova 2017), (Pisano, Capecchi 2015), (Pisano, Fichant, Bussotti, Oliveira 2017), (Gillispie, Pisano 2014, chapters 2-5).

In order to establish a dynamical justification to field equations, Maxwell focused on the fact that magnetic field appears as a completely kinetic system whose energy should be purely kinetic. Therefore, the Lagrange's equations are expressed in terms of a kinetic energy T, which depends on the variables q_i and on velocities. In particular, the dependence of T on the variable q_i, that is dT/dq_i, is not only a mathematical derivative operation but it represents a physical variation of q_i.

In other words, the state of equilibrium of a system is unchanged while q_i is varied (Simpson 2005, p. 65). This is a sort of thought-experiment. By differentiating the energy function by using Lagrange's equations of motion, Maxwell formulated his *momentum concept* (Maxwell 1873, II, part IV, chapter V, pp. 190-193), which is the basic concept to build in his dynamical theory (Maxwell 1873, II, part IV, chapter VI, pp. 195-205). Therefore, Maxwell formulated his mathematical field concept through three phases:

a. a geometrical study of Faraday's hydrodynamic analogical model, based on systems of lines of forces imagined – by Faraday – "the collection of imaginary properties" (Maxwell 1856, p. 160, line 4) of the theory of motion of an incompressible fluid;
b. a concrete mechanical model (Maxwell 1861-1862, part I), based on the production of magnetic forces from electric current, called molecular vortices;
c. a *dynamical justification* to field equations (Maxwell 1865, 1873).

Following these three steps Maxwell built his dynamic theory presented in the sixth chapter of his *Treatise* (Maxwell 1873, II, part IV, chapter VI, pp. 195-205) where the equations were proposed to bear on a simple electromagnetic system consisting of only two circuits.

At the end of chapter five, Maxwell emphasizes that in his review of Lagrange's method it took no account of the mechanism by which the parts of a connected system interact with each other, as stated at the beginning of the chapter. Furthermore, the author underlines the importance of the development of purely mathematical methods in building a new theory of dynamics, because this approach gives the possibility that some unknown truths come out. Once again a strict connection between physics and mathematics emerges (Maxwell 1873, III, part IV, chapter V, pp. 193-194).

4. Concluding Remarks

Following the three steps described above, Maxwell's method can be extended to more complex systems involving many conductors and mechanical motions and the corresponding equations are interpreted in electrical terms following Faraday's experimental discoveries on electromagnetic induction (Pisano 2013b, 2017; Darrigol 2005). In Chapter VIII (Maxwell 1873, part II, chapter VIII) Maxwell worked out the mathematical structure of the electromagnetic field making wide use of Faraday's studies and experimental results.

Finally, in order to build a new dynamical theory, Maxwell needed something unlike Newtonian mathematics (Pisano, Bussotti 2015a, 2016). Lagrange's mechanics, such as a new approach to physical theory through the idea of a connected mechanical system (expressed in Lagrange's equations), appeared to be the solution.

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