Dirac's quantization improved by Morchio & Strocchi to an algebraic structure founding both classical mechanics and quantum mechanics

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Abstract: It is well-known that Dirac suggested a quantization of Classical Mechanics by means of an analogy between classical Poisson brackets and commutation relations. Morchio and Strocchi suggested a rigorous derivation of this quantization by finding out, independently from Dirac's previous works, a new algebraic structure which characterizes both kinds of Mechanics as two representations of this structure, distinguished by a dichotomous variable *Z,* whose value 0 represents the case of Classical Mechanics, whereas the value $ih/2\pi$ represents Quantum Mechanics. No longer Classical Mechanics can be considered as the limit of Quantum Mechanics for $h \to 0$; and these theories have to be considered as mutually incommensurable. The nature of this incommensurability is investigated; in particular, it is compared with the incommensurability between two formulations of classical Mechanics, i.e. Newton's and Lazare Carnot's.

Keywords: Quantum Mechanics, Dirac's quantization, New algebraic structure of Quantum Mechanics, Incommensurability of two divergent Mechanics, Problem-based theory.

1. Introduction

Dirac had an advantage with respect to most theoretical physicists; he well-knew Hamilton formulation of Classical Mechanics (CM), which at his time was undervalued and put aside. (Jammer 1989, p. 236) It enjoys extraordinary properties. Equipped with canonical variables the Hamiltonian is independent from the system of reference. Moreover, its basic operation is the Poisson brackets (PB), through which this formulation translates all basic operations of calculus; e.g. the derivative of a physical magnitude *k*, the case of time included, is equal to a PB of the Hamiltonian *H* and this magnitude *k* with respect to the two variables of the phase space; hence all differential operations of Hamiltonian dynamics are represented by an algebraic structure of the PBs. In other terms, no other formulation of theoretical physics enjoys a strong link between physics and mathematics as Hamiltonian formulation through its PB algebraic structure. In addition, it is remarkable that this link concerns what is more advanced in both branches of science, i.e. Mechanics and Mathematics. This fact suggests to attribute to the Hamiltonian a leading role in advancing theoretical physics. One should investigate why in the history of Theoretical physics this great theoretical advancement was undervalued before Dirac's application to QM. Rightly Strocchi resumed the historical development of Analytical Mechanics in a new way, i.e. as addressed to both achieve and develop the Hamiltonian formulation. (Strocchi 2018)

In 1925 Dirac discovered an analogy between the Poisson brackets of Classical Mechanics (CM) and the quantum relations of commutation. Through this analogy he suggested an algebraic structure characterizing Quantum Mechanics (QM). Being his analogy between QM and CM very effective in order to suggest great part of the new formalism, he looked for improving it into a more substantial homeomorphism. Yet, it eventually resulted to be formally inconsistent (Darrigol 1992, Chapt. 12; Ali, Englis 2005, Sect. 1).

After a long debate on how improving this method of quantization (Landsmann 2005), recently two papers (Morchio, Strocchi 2008, Morchio & Strocchi 2009) – both summarized by a paper (Strocchi 2012), and chapter 7 of a book (Strocchi 2018) – revisited Dirac's quantization by suggesting a specific algebraic structure, which gives a complete solution of the quantization problem in terms of a dichotomous variable Z – the case $Z = 0$ represents CM, whereas the case of an imaginary value of Z, QM – and moreover gives reason of the partial failure of Dirac's quantization.

2. Morchio and Strocchi's algebraic structure accurately representing the problem of quantization

When introducing his strategy, Strocchi underlines the relevance of the algebraic structure of the classical Hamiltonian:

> […] the algebra of canonical variables with the (Lie) product defined by the Poisson bracket provides the general structure for the formulation of Hamiltonian classical mechanics and may be considered as its backbone. Actually, most of the general issues, like time evolution, transformations of canonical variables, symmetries and constants of motion etc. have a simple and neat formulation in terms of such an algebraic structure. Clearly, Dirac must have had in mind the power and effectiveness of the classical canonical structure in formulating the quantization rules in such a way to reproduce as closely as possible the general algebraic properties of Hamiltonian mechanics (*canonical quantization*).
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> In fact, in this way, important physical quantities, like e. g. the Hamiltonian, the momentum and the angular momentum keep being the generators of, respectively, time translations, space translations and space rotations, provided that their action is given by commutators rather than by the Poisson brackets.
>
> Amazingly, as it may *a posteriori* appear, at a *formal level* the quantum revolution may be reduced and fully accounted for, merely by the replacement of the classical Poisson brackets {. , .} by commutators (*Dirac canonical quantization*) […]
>
> $[\boldsymbol{q}_i, \boldsymbol{q}_j] = 0 = [\boldsymbol{p}_i, \boldsymbol{p}_j]\ \ldots\ [\boldsymbol{q}_i, \boldsymbol{p}_j] = ih/2\pi\{q_{i,}, p_j\} = ih/2\pi\ \delta_{ij}$

where [. , .] denotes the commutator and ***q***, ***p*** the quantum version of the classical canonical variables *q, p*.

The issue of further understanding and justifying such a strong relation between classical and quantum mechanics was of great concern for Dirac.

Dirac suggested to explain the above relation between classical and quantum mechanics by abstracting [what then Morchio and Strocchi recognized to be] the following algebraic structure as common to both classical and quantum mechanics…. The [only] very important property discovered by Dirac is that in a Poisson algebra[1] $\mathcal{A}$ the following algebraic identity (*Dirac identity*) holds, which establishes [Dirac quantization, i.e.] a link between the Lie product { , } and the commutator $[A, B] \equiv A\,B - B\,A$ (defined in terms of the basic product), for any $A, B \in \mathcal{A}$.

Proposition 7.1.1 *If $\mathcal{A}$ is a Poisson algebra [of real, regular functions] the following [six] properties hold*

1) (*Dirac identity*)

$$[A, C]\,\{B, D\} = \{A, C\}\,[B, D]; \qquad \forall\, A, B, C, D \in \mathcal{A}; \tag{7.4}$$

2) *the commutator and the Lie product commute*

$$[A, B]\,\{A, B\} = \{A, B\}\,[A, B]; \tag{7.5}$$

3) *if there exists a pair C, D, such that {C, D} is invertible, as assumed in the following, then*,

$$[[C, D]\,,\{C, D\}^{-1}] = 0$$

$$[[C, D]\,\{C, D\}^{-1}, \{A, B\}] = 0 \qquad \forall\, A, B \in \mathcal{A}; \tag{7.6}$$

4) *if also {A, B} is invertible, then*

$$[A.\,B]\,\{A, B\}^{-1} = [C, D]\{C, D\}^{-1} = \{C, D\}^{-1}[C, D] \equiv Z, \tag{7.7}$$

5) *Z relates the commutator to the Lie product, in the sense that* $\forall\, E, F, G, H \in \mathcal{A}$

$$[E, F] = Z\{E, F\}, \quad [Z, \{G, H\}] = 0 = [Z, [G, H]]. \tag{7.8}$$

6) *Z commutes with all the elements of A, i.e. it is a central variable:*

[1] Let us recall that an *algebra* is a vector space over the complex numbers with an associative product which is linear over both factors. The *Lie product* of two functions is given by their PB, enjoying the properties of antisymmetry, linearity on both factors, Leibniz rule and Jacobi identity. The *inverse* of a PB is obtained by exchanging the derivation variables with the functions to be derived; they are called Lagrange brackets, the first kind of brackets introduced into theoretical physics. A *Poisson algebra* is a real associative algebra equipped with a Lie product.

$[Z, A] = 0 \qquad \forall A \in \mathcal{A}.$ (7.9)

> […] Actually, eq.s (7.8) and (7.9) do not hold for general Poisson algebras. In particular, the existence of pairs C, D such that {C, D} is invertible fails if the Poisson algebra is generated by C^∞ functions of compact support (Strocchi 2018, pp. 93-97).

Moreover, the corresponding algebraic relation between classical and quantum canonical variables results incompatible with some valid rules (e.g. linearity). The long time research on this subject led to a "quagmire" of contradictions among the several requirements for satisfying a suitable quantization (Ali, Englis 2005, p. 397).

> In conclusion Dirac argument for explaining the deep relation between classical and quantum mechanics at the level of the canonical structure is mathematically inconsistent and the Lie algebraic structure of the quantum variables cannot be defined [according to Dirac's suggestion] (Strocchi 2018, p. 98).

3. Morchio and Strocchi's new algebraic structure for both kinds of Mechanics

After having put the problem, Morchio and Strocchi illustrate the basic point of their solution.

> Our results suggest a quite different approach to the relation between classical and quantum mechanics with respect to phase space quantization: *the classical phase space is not assumed* [emphasis added] as a starting point [as Dirac did it] and rather arises [as obtained from the positions q_i only] from the same (non commutative) Poisson algebra [defined by the Proposition 7.1.1 in the above], in correspondence with one of the values taken by the variable *Z*, on the same footing as the quantum mechanical phase space (Morchio, Strocchi 2009, p. 38).

The momenta p_i are generated by the derivative operator of the positions q_i; inside Hamiltonian mechanics it corresponds to a PB. Then their free polynomial associative (real) algebra is generated. This algebra, together with the same kind of algebra of the positions q_i constitutes the algebra $\mathcal{A}$ of these coordinates. This point is very important; here the classical coordinates phase space is a result, whereas the algebraic relations of PBs is basic; hence, the algebra is put first, then comes the functions space.

Then the Lie product is defined in such a way to assure – through the Rinehart definition of its extension to vector fields – the functional correspondence between the algebra $\mathcal{A}$ and the vector space Vect($\mathcal{M}$) [the module structure of the Lie algebra of vector fields of compact support] to the Vect($\mathcal{M}$). Eventually, a Poisson-Rinehart algebra is thus defined as the framework of the theory.

In such an algebra Proposition 7.1.1 is proved to be true by (Strocchi 2018, pp. 96-97). Therefore, a central variable Z is obtained from the algebraic relations between commutators and PBs.

> In particular, the above construction shows that the Dirac *ansatz* of canonical quantization, in the form of the proportionality of the commutators of variables in $C^\infty(\mathcal{M})$+ Vect($\mathcal{M}$) to their classical Poisson brackets, has no alternative, within the above rather general notion of mechanical system.
>
> [CM] arises from the same (non-commutative) Poisson algebra in correspondence with one of the values taken by the central variable Z, on the same footing as the quantum mechanical state space. We also emphasize that in the above approach the Planck constant needs not to be introduced. It automatically appears as a variable invariant under all physical transformations, i.e. a universal constant, in the Poisson—Rinehart algebra of a manifold (Morchio, Strocchi 2009, p. 38).

Strocchi concludes:

> The intrinsic geometry of the algebraic structure of $\mathcal{A}$, namely the existence of the translations in the space of coordinates, replaces the somewhat *ad hoc* assumption [suggested] by Dirac… on the basis of a claimed classical analogy. No such a classical analogy may be invoked for relating classical and quantum mechanics; the only relation is that they correspond to *inequivalent* realizations of the Poisson algebra $\mathcal{A}$ defined above, which consists of free polynomials[2] of the coordinates q_i and the generators of translations q_i.
>
> The inequivalence of the two realizations precludes the existence of a mapping between them (as for inequivalent realizations of a Lie algebra) and explains the obstructions for Dirac strategy (Stocchi 2017, p.100).

The above results constitute an independent approach to the quantization; it is at all independent of its historical origin in Dirac's analogy.

4. The mutual incommensurability of CM and QM

In my opinion, this result cannot be underestimated for several reasons. *First*, we have the same mathematical formalism for both CM and QM, differing only in the value of a discrete parameter. No other pair of physical theories enjoys such a property within the almost four centuries long historical development of theoretical physics, i.e. within the most advanced field of theoretical science.

Second, this formal convergence in describing two very different realms – the macroscopic one and the microscopic one – means that this algebraic structure represents a fundamental structure of our representation of the world. No philosopher of science is allowed to ignore this advancement.

Third, the variable Z is dichotomous. Hence, the traditional considerations on the translation of QM in CM by means of a limit operation for $h \rightarrow 0$ are of a merely intuitive nature (as all the limit operations on physical situations which are not entirely

[2] In a first approximation, the adjective "free" may be here intended as "generic".

translated into mathematical terms); $h \to 0$ recovers commutativity, but for instance the canonical structure is lost (Strocchi 2012, p. 10). Notice that in the past some authors questioned such a limit, yet without giving indisputable evidence for proving its inaccuracy (Bokulich 2010).

Fourth, Morchio and Strocchi's algebra is the first mathematical structure which accurately distinguishes two distinct physical theories, CM and QM, as two inequivalent representations of this algebraic structure. No stronger evidence is possible for concluding that the two physical theories are mutually incommensurable (by taking the last word in an intuitive sense[3]). Notice that two theories manifest their incommensurability through radical variations in meaning of their common notions. In the history of theoretical physics, the notions shared by physicists no more changed than by the introduction of quantum notions; it is enough to recall the quantum notion of wave-particle with respect to the distinct meanings of the two classical notions of either wave or particle (Dirac's book begins just by underlining this point; Dirac 1930, Chapt. 1, Sect. 1). In addition, Morchio and Strocchi's algebra gives the first instance of a pair of incommensurable theories whose incommensurability is expressed in mathematical terms. This case ends the long-time debate whether this philosophical notion has only speculative import or rather has a relevant role inside the foundations of science (Oberheim, Hoyningen-Huene 2018).

Fifth, the variable Z is of a dichotomous nature, as the basic dichotomies of the foundations of science (Drago 1988; Drago 2017); however, the former one is a numerical variable and the latter ones range on theories (respectively, the kinds of mathematics and the kinds of logic). I suggest that the variable Z corresponds to the dichotomy problematic organization/axiomatic organization (PO/AO) for the following reason. Let us recall that the introduction of imaginary numbers into the field of real numbers applies the philosophy pertaining to a PO theory. The imaginary numbers represent a non-standard model of the system of real numbers; this model is built as a hypothesis on the system at issue which makes use of only real numbers; this hypothesis is elaborated in order to eventually obtain a concrete result in real numbers, to be then compared with the given system. As an instance of this philosophy as applied inside a mathematical theory is Lobachevsky non-standard model of Euclidean geometry of the usual model of Euclidean geometry. He obtained it by just replacing inside spherical trigonometry the angle α by the correspondent imaginary angle $i\alpha$. He called his new model exactly "imaginary geometry". Also in theoretical physics the ordinary QM introduces imaginary numbers (for representing the amplitude of probability) according to the same philosophy, being the result of this introduction real numbers to be compared with the real numbers obtained from measurements. In conclusion, owing to this philosophy, the imaginary numbers always introduce a PO into both mathematical and physical theories.

Sixth, the case of $Z = ih/2\pi$ denotes the algebraic structure of respectively the Hamiltonian QM, which has to be considered a PO theory owing to the imaginary numbers, and the case $Z = 0$ the Hamiltonian CM, which is an AO theory; equivalently, the latter me-

[3] The most celebrated pairs of theories are, on one hand, CM and, on the other hand, one of either QM or special Relativity.

chanics is governed by classical logic, whereas the former one by intuitionist logic. This alternative on the kinds of logic agrees with a previous result, i.e. QM makes essential use of intuitionist logic (Drago, Venezia 2002).[4] This result is at odd with common studies in quantum logic because, in order to not deny classical logic as governing the entire theoretical physics, the scholars assume a local viewpoint (Jammer 1974, sect. 8.1). Instead, according to my viewpoint all the physical theories, both classical and non-classical, are severed by the choice on the kind of logic and QM chooses the same intuitionist logic (governing a PO formulation) which a lot of theories already chose: Lazare Carnot's mechanics, Sadi Carnot' thermodynamics, Einstein's special relativity, Einstein's 1905 paper on quanta and, last but not least, Dirac's book.

Seventh, the most accurate definition of incommensurability is the structural one, which is given by a difference between the two pairs of theories in comparison.[5] Within the history of classical physics the most relevant incommensurability case is represented by the two following formulations of mechanics: Newton's one, whose basic choices are AI&AO, and Lazare Carnot's mechanics, whose basic choices are PI&PO.[6] The logic of the former one is classical, whereas the logic of the latter is intuitionist (this theory looks for the in-variants of the collisions). Moreover, their theoretical development are at odd. Whereas the first step of the former is statics and then comes dynamics, the first step of the latter is dynamics and then, at the equilibrium, comes statics. The same occurs in the above cases of the Hamiltonian; the first step of classical Hamiltonian is to state the set of all trajectories as summarized by its two characteristics, the first order differential equations, whereas the first step of quantum Hamiltonian is to state the dynamics determined by the relations of commutations.

E*ighth*. Whereas Newton's formulation makes an essential use of calculus, L. Carnot's formulation makes use of no more than algebraic-trigonometric mathematics (Drago 2004). Actually, for a long time the latter formulation was the only formulation of Mechanics making use of an algebraic formalism (and probably also for this reason it was depreciated for a long time). 140 years passed before Dirac, through his analogy, re-introduced a modern algebraic formalism inside a formulation of Mechanics, and 210 years passed before Morchio and Strocchi improved this analogy into an accurate formalism which eventually was put as the very foundations of the dynamics of the two formulations of Mechanics, mainly of QM.

Ninth. Their incommensurability may also be represented by the radical variation of a physical model, the ideal model of bodies collision; either perfectly hard bodies whose shapes are invariant and hence the total energy is not conserved; or perfectly elastic bodies, which behave as springs and hence their total energy is conserved. This difference is dichotomous in the conservation of total energy, ΔE_{tot} being zero in the

[4] This result is corroborated by an analysis of Dirac' book, which as a fact illustrates QM through an essential use of doubly negated propositions of intuitionist logic.

[5] Several pairs of physical theories are mutually incommensurable; in the historical development of theoretical physics the first pair of such theories was Descartes' optics and Newton's optics (Drago, Guerriera 1986), the most important pair of classical physics was Newton's mechanics and Lazare Carnot's mechanics (Drago 1988) (and also Sadi Carnot's thermodynamics; Drago Pisano 2000).

[6] AI: Mathematics with actual infinity; PI: Mathematics with potential infinity.

latter case or not in the former case. This dichotomous formula is more complex than that of the above parameter *Z*, distinguishing CM from QM; however, it is very near.

Tenth, already some scholars (Kronz, Lepher 2012) intuitively recognized a contrast between von Neumann's approach to QM and Dirac's approach, contrast which persisted within subsequent theoretical results, i.e. the contrast between Wightman's Axiomatic Quantum Field Theory and the Algebraic Field Theory. By qualifying in mathematical terms Dirac's approach, Strocchi and Morchio's result makes mathematically accurate the former contrast. This result allows to qualify in mathematical terms also the contrast within the theoretical development after QM.

Some problems are left open: 1) To specify the AO in the Hamilton classical formulation of mechanics, built by means of the algebraic structure of PB. 2) To formally derive from Morchio and Strocchi's algebra of QM a lattice which of course represents a non-classical logic, which ought to be the intuitionist logic. 3) To find out the counter-part of this algebra in constructive mathematics.

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