A dozen formulations of quantum mechanics: a mutual comparison according to several criteria

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Abstract: In the literature there exist more than a dozen formulations of quantum mechanics, their number being different according to what one means for ‘formulation’. Scholars have suggested some interpretative categories of this variety of formulations; these categories are compared to the two dichotomies – on the kind of infinity and the kind of a theory organization – which are here considered as the foundation of a scientific theory. A general interpretative framework is obtained. By means of all these categories the above mentioned formulations are classed. Most of these formulations share the Newtonian choices for the actual infinity and the deductive organization. The few formulations based on the alternative choices are recognized; they include the first one, i.e. Heinseberg’s, and the recent Bub' one.

Keywords: Quantum mechanics. Formulations. Interpretative categories of formulations. Basic dichotomies. Non-deductive organization. Intuitionist logic. Constructive mathematics,

1. The basic ambiguity of the present theory of quantum mechanics

In the following I will investigate on the foundations of theoretical physics, by avoiding to deal with contemporary philosophy of quantum mechanics which someone evaluates as a ‘much-malignant discipline’. (Jagannathan 2002, p. 1272)

No physical theory in the past was so accurately adequate to the reality as quantum mechanics (QM). Indeed, no discrepancy between a quantum phenomenon and a theoretical prevision is detected. Moreover, notwithstanding representing a radical philosophical change with respect to classical theories, QM resulted to be a stable theory along almost a century. Yet, several scholars were unsatisfied of its theoretical construction. In particular Albert Einstein has launched the celebrated attack: ‘God does not play to dice’. As an authoritative scholar puts it: “According to Feynman, nobody really understands quantum mechanics. By this I think he means that nobody understands why nature has chosen to compute probabilities in a so strange way”. (Gudder 1988, p. XII) Some year later he added:

> Although quantum mechanics is over 90-year old, it still contains many perplexing mysteries. As evidence for the dissatisfaction with the subject, there are […] [several] approaches to the foundations of quantum mechanics […] Why are

> researchers in this field so discontent that they are continually manipulating its fundamental axioms? There are several reasons for the present state of flux. Although quantum mechanics has been eminently successful and has made many correct and precise predictions, we still lack a deep understanding of its foundations. Quantum mechanics, as it now stands, consists of a cookbook of seemingly *ad hoc* rules and recipes. We do not really understand where these rules come from and why they work, but must simply accept them on blind faith. If progress has to be made, we must obtain a deeper grasp of the subject.
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> […] because of its lack of a rigorous foundation quantum mechanics has its logical problems as demonstrated by the plague of infinities and divergences in quantum fields theory. Quantum mechanics at present cannot adequately explain and describe the plethora of "elementary" particles, nor has a successful theory of quantum gravity born developed. It must be granted that towards these ends, quantum chromodynamics, quantum gauge theory, and superstrings theory are being intensely pursued. However, despite these efforts, these theories have exhibited very little predictive power. (Gudder 1992, pp. 15-16)

One may add some more unsatisfactory aspects of this so admirable theoretical construction.

1. Likely the theory of mechanics, this theory may be formulated in several ways, but this variety of formulations is disconnected from the variety of formulations of the previous theories. In particular, classical physics includes some formulations which are well-known alternatives – in the sense that the most basic notions and techniques are the opposite ones – to the dominant ones. Indeed, there exists a clear alternative formulation to Cartesian optics – i.e. in a first time Christian Huygens' one and the Clerk Maxwell's one –; there exists a clear formulation which is an alternative also to Isaac Newton's mechanics – i.e. Gottfried Leibniz–Lazare Carnot's one (Drago 2004) –; there exists a clear alternative formulation to the phenomenological thermodynamics – i.e. Constantin Carathéodory's one. Yet, in QM no alternative formulation to the dominant one (Schroedinger-Dirac-Von Neumann in Hilbert space) of my attempts is recognised.[1]
2. Since the logical laws are at variance in the different kinds of logic, no theoretical physicist has to ignore to which logic he is obeying. Yet, the question of which kind of logic is appropriate to QM was investigated only after the accomplishment of the theory. In the year 1936 QM was declared the first physical theory governed by a non-classical logic (Birkhoff, von Neumann 1936); yet, this great novelty did not lead to discover a corresponding formulation of QM.

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[1] Actually QM born through Heisenberg's matrix mechanics, which for a short time remained the alternative one to Schroedinger wave mechanics [On this episode see the accounts by Beller (1983) and Giannetto (1997)]. Yet, both have been considered as included in the most general formalism of the at present dominant theory.

3. The mathematics of quanta is of course discrete; yet, the dominant formulation of QM is based on a continuous framework prolonging the traditional mathematical physics. From the viewpoint of the dominant formulation all seems to have passed as if the unexpected discovery of the quanta was a path accident, which had to be repaired as soon as possible. Instead, in 20th Century Mathematics the discrete approach was widely developed and formalized (Bishop 1967) and new scientific theories have built according to it (one for all, computer theory).
4. Although the indeterminacy principle is the essential reason for introducing a new theory with respect to the classical physics, most textbooks locate it at the end of the theoretical development of QM, as if it was a measurement question only. In this way Janos von Neumann's paradox – i.e. the theory is composed by two parts which are mutually incompatible; one part concerning the unperturbed system and the other part the perturbed system (Drago 1991b) – is kept away in the far horizon.
5. Erwin Schroedinger, Paul Dirac and Janos von Neumann formulated QM according to the traditional mathematical technique, i.e. the differential equations. Their mathematical framework, Hilbert space, is an *a priori*, abstract framework claiming to obtain a mathematical omniscience. An alternative mathematical technique is symmetry. (Barut 1986) Yet, since the year 1925 Hermann Weyl tried unsuccessfully to build QM by means of symmetries. (Weyl 1928; Drago 2000) Eventually in the '60s, the symmetry technique was no more considered a "pest" and hence was widely used by theoretical physicists as much as, if not more than differential equations. However a formulation of quantum mechanics directly based on symmetries is still lacking.
6. By coming back from QM to classical mechanics through the limit $h \rightarrow 0$ one obtains the Hamilton-Jacobi formulation only. This fact proves that the various formulations of a same physical theory – in this case, classical mechanics – are inequivalent with respect to the kind of mathematics and hence the kind of language. Thus, in order to understand QM by starting from classical physics, a theoretical physicist would have to a priori decide which formulations of respectively classical mechanics, electromagnetism and thermodynamics are his basic ones; otherwise, the various notions about which he argues, being referred to inequivalent theories suffer radical variations in their meanings.
7. I proved (Drago 1986; Drago 1996) that according to the constructive mathematics – i.e. the mathematics rejecting the axioms appealing to the actual infinity; for ex., Zermelo's axiom – Newton's mechanics is undecidable. Then it was shown by da Costa and Doria (1991) that also the Hamiltonian mechanics is undecidable. Although one is allowed to suspect that the principles of the dominant formulation of QM – appealing to actual infinity – constitute an idealistic framework, no specific theorem was proved

as undecidable. (Billinge 1997) On the other hand, a re-formulation of QM through constructive mathematics is still lacking.

2. A dozen formulations of quantum mechanics

Eighty years of collective pondering on QM produced a variety of formulations – or even pictures. A recent paper listed a dozen formulations of QM, (Styer *et al.* 2002)[2] chosen as those formulations which may be interesting to a working theoretical physicist:

> Each of these formulations can make some application easier or some facet of theory more lucid, but [one has to take in account that] no formulation produces a royal road to quantum mechanics. (Styer *et al.* 2002, p. 195, I)

They are the following ones:

1. The matrix formulation (Heisenberg);
2. The wave function formulation (Schroedinger);
3. The path integral formulation (Feynman);
4. The phase space formulation (Wigner);
5. The density matrix formulation;
6. The second quantization formulation;
7. The variational formulation;
8. The pilot-wave formulation (De Broglie-Bohm);
9. The Hamilton-Jacobi formulation.

For each of them the paper specifies:

I. the mathematical formalism,
II. its application to the case-studies of either one or two, or infinite particles (either bosons or fermions),
III. a quick note on its history,
IV. a succinct list of original references.

In a final section (‘Additional issues’) two more ‘interpretations’-formulations are considered:

1. The many-worlds formulation (Everett);
2. The transactional interpretation (Cramer); Eventually, three ‘miscellaneous issues”’are declared not properly formulations;
3. The density functional theory;

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[2] Previously the same review published two *Resource Letters on the various aspects of Quantum mechanics.* (DeWitt, Graham 1971; Ballentine 1987)

4. The consistent histories;
5. Continuous spontaneous localization.

Some more formulations may be suggested. ‘Physics Stock Exchange’ adds to the previous ones the new ones: 15. PT symmetric quantum mechanics; 16. Superoperator formulation. I add: 17. Strocchi’s based on C* algebra. (Strocchi 2005)[3]

3. The categories for classifying the various formulations

The motivations of the authors of such formulations range from a radical operativism – “Shut up and calculate!” (Mermin 1989, p. 9) – to the introduction of physicist’s consciousness or even the application of the anthropic principle. Since these extreme motivations are of a philosophical nature – although presented in physical clothes –; their examination involves to consider both theoretical and philosophical aspects.

Some scholars tried to class them according some foundational differences. Wikipedia article is interesting also because it classes the listed formulations according to the following 9 questions: Determinist? Wavefunction real? Unique history? Hidden variables? Collapsing wavefunctions? Observer role? Local? Counterfactual definiteness? Universal wavefunction exists? Notice that the nature of these questions are not only technical, but also philosophical.

It is well-known that Einstein suggested two categories for analysing a formulation of QM: Realism and Completeness. To them the subsequent debate on the foundations of QM added: Local realism and Determinism. I will call them ‘Einstein’s categories’. I define these four categories as best I can through the definitions offered by the current literature.

Completeness: “No theoretical construction can yield experimentally verifiable predictions about atomic phenomena that cannot be extracted from a quantum theoretical description.” (Bohr and Heisenberg)

Determinism: “A theory is deterministic if, and only if, given its state variables for some initial period, the theory logically determines a unique set of values for those variables for any other period.” (Nagel 1999, p. 292) Alternatively: For every event, including human action, there exist conditions that could cause no other event (“Interpretations of quantum Mechanics” Wikipedia 2016).

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[3] In Wikipedia an anonymous article ([online]. <http://en.wikipedia.org/wiki/Interpretations_of_quantum_mechanics> [data di accesso: 12/05/2016].) lists 18 *interpretations* of “both the formalism and the phenomenology” of QM, each described in few lines. Since the definitions of “interpretation” and “formulation” are not the same, one has to expect that the new list differs from the former one. Indeed, it does not include eight of the previous formulations: 1, 3-7, 9 and 12 whereas it adds the following ones interpretations: 18. Ensemble interpretation: 19. Relational quantum mechanics; 20. Elementary cycles; 21. Transactional interpretation; 22. Stochastic mechanics; 23. Von Neumann-Wigner consciousness; 24. Participatory anthropic principle; 25. Many minds; 26. Quantum logic; 27. Quantum information theory; 28. Modal interpretations of quantum theory; 29. Time symmetric theories.

Realism: "The material objects exist in themselves, apart from the mind's consciousness of them". (Heisenberg 1958, p. 100) The minimal realism: Any scientific theory should be interpretable as a mind-independent description of the world. (Johansson 2007, p. 15)

Local realism: This principle is the combination of the principle of locality with the 'realistic' assumption that all objects must objectively have a pre-existing value for any possible measurement before the measurement is made. The principle of locality is defined as follows: There is no way that two systems can interact with each other 'instantaneously' at a distance (i.e. faster than light). An object is influenced directly only by its immediate surroundings [through actions operatively determinable] (Interpretations of ...); One more definition is the following one: A local realist theory is one where physical properties [of microscopic world] are defined prior to and independently of measurement, and no physical influence can propagate faster than the speed of light. In other words, measurements do not betray us.

A further suggestion came from (Caponigro 2010, chp. IV). He suggested to class the formulations according to two dichotomic categories, i.e. the different ways allowed to "explain the observer and the underlying physical reality once established at ontic level": Realism/Idealism, Ontic/Epistemologic. By considering them as a two-dimensional diagram, several formulations are represented as points aligned along a straight line.

4. The categories of the two basic dichotomies. The mutual comparison of all the above interpretative categories

In previous papers I presented two dichotomies as constituting the foundations of theoretical physics: 1) The dichotomy on the two kinds of infinity in mathematics, or evenly, the formal dichotomy of either the classical mathematics freely appealing to the actual infinity (AI), or the constructive mathematics based on the potential infinity only (PI). 2) The dichotomy on the kinds of the theory organization – either the deductive organization (AO), or the problem-based organization (PO) whose ideal model was defined by a previous paper –, or evenly, the formal dichotomy of either the classical logic or the intuitionist logic. Two theories differing in their basic couples of choices, are defined as incommensurable. (Drago 1991a; Drago 2014)

Can the above mentioned two dichotomies interpret and summarize the previous multitude of categories? A strategy may be to perform a specific analysis of each formulation in order to decide its basic choices; then to compare it with the above categories. But this analysis is very difficult, since may involve all the theoretical aspects of a formulation; let us recall that to decide the idealistic nature – the appeal to AI in the basic notions and techniques – of Newton's mechanics required a long time and even more time was required for deciding the idealistic nature of its mathematics (chaos, undecidabilities). Drago 1986; Da Costa Doria 1991) Hence, the following determinations have to be considered as a work in progress. An alternative strategy is to compare directly the two dichotomies with the above categories. Even this comparison is difficult for two reasons. First, the categories – for instance, Einstein's four ones – are variously defined. Second, the physicists suggested categories through notions which almost all belong to the objective realm; a sharp correspondence between them and the two dichotomies, which instead refer to the structure of as theory, is not easy. However this strategy constitutes at least as a first attempt. However, no surprise if the correspondences suggested in the following are partially founded and not sharply defined.

The choice PO is of a global nature; i.e. it implies the capability to consider at once all the phenomena useful to the solution of its basic problem; hence it means 'Completeness' in a sense very similar to the common definition in QM.

A deductive theory, i.e. a theory choosing AO, is entirely determined by few axioms developed according to classical logic; hence the 'Determinism' may be considered as an allusion to AO.

The choice PI means in physical terms the choice of operativism; hence it may be intended as alluded by the *Realism*, as previously defined.

The choice AI may be intended as physicists' capability to fully determine the properties of the infinitely small, hence the capability to manage them through a mathematics relying on the AI; in fact, in the dominant formulation of QM the choice AI includes Dirac's delta – i.e. the adjunction of an ideal mathematical element for rounding the basic notions – and Hilbert's space – whose functions include also the most sophisticated ones. Hence, the mathematical appeal of AI corresponds to the 'Local realism'.

The correspondence with Caponigro's dichotomic categories seems equivalent to te previous ones. Provided that Idealism is intended as mathematical idealism, his 'Realism and Idealism' may be intended as respectively PI and AI. The correspondence between Ontic/Epistemologic with AO/PO is loose, because the former notions are of philosophical nature only. Yet, a sharp correspondence appears when the Ontic is intended as the Newton's one, i.e. a theory deduced (AO) from metaphysically certain principles; and the Epistemologic is intended at a methodological level, just what is a theory PO, aimed to find out a new method for solving a general problem. In sum, the correspondences are the same of the Einstein's four, apart the different coupling two by two of then four categories.

Two of the latter ones of the 9 Wikipedia questions concerns the same subject of the two of Einstein's categories – Deterministic, Local [realism] – and hence their correspondence of the two basic dichotomies with the choices AO and AI have been already established. It is difficult to establish a correspondence between the remaining two choices PO and PI and Wikipedia questions because the latter ones correspond to a manifestly different viewpoint from that of the basic choices. The result is the following one (In Italic are the attributions which I consider as certain): *Determinist*: AO. *Wavefunction real*: AI, since it attributes a reality to unknown beings. Unique history (PI*?*). *Hidden variables:* PI, inasmuch as it wants to recuperate the operativism of classical physics (or also AI?). Collapsing wavefunctions: AI (?). *Observer's role*: it destroys the idealistic illustration offered by an AO theory, hence, PO. *Local* (realism): AI. Counterfactual definiteness: PO (vs. AO?). *Universal wavefunction exists*: AO.

The following table summarises the above attributions.

Basic choices	AI	PI	AO	PO
Einstein's categories	Idealistic mathematics (Local realism)	Operativism (Realism)	Determinism	Completeness
Wikipedia's classification criteria	Local, Real wavefunction, Collapsing wavefunction	Hidden variables, Unique history	Determinist, Existence of a universal wavefunction	Observer's role Counterfactual determination

Table 1. Mutual comparison of three kinds of Categories

5. Applying a foundational viewpoint in order to classify the various formulations

The previous correspondences help the task of attributing to each formulation of QM its dichotomic choices. This attribution is decided on the basis of the direct observation on its most apparent mathematical and organizational aspects, its summary description in the literature and mainly its classifications in both Wikipedia and Caponigro's classifications of all the formulations according their categories. I take in account the formulations of QM listed by Styer *et al*., yet I add Bub's formulation (Bub 2005) since it is a recent instance of formulation based on the alternative choices.

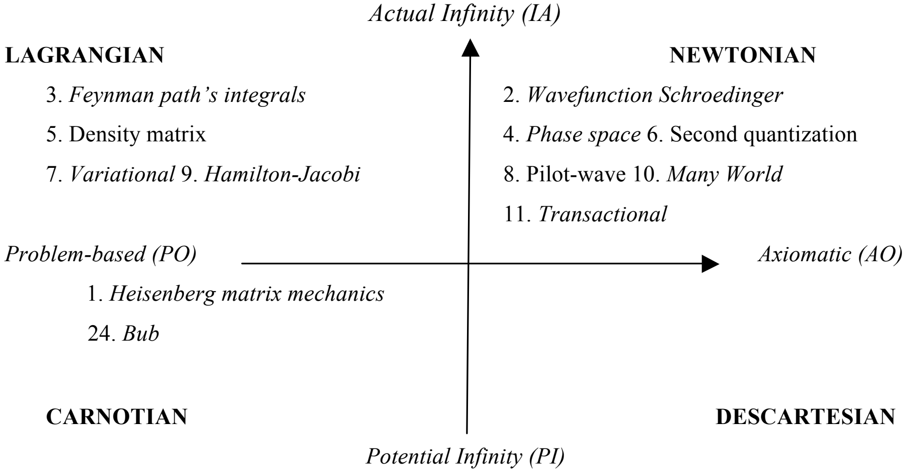
IA: 2., 3. (integral of all paths), 4. (idealistic phase space distribution), 5. (?), 6. (idealistic creation and annihilations), 7., 8., 9., 10., 11..

IP: 1., 24., 25..

AO: 2., 4. (a priori phase space distribution), 5. (a priori density matrix), 6. (?), 8. (causality), 10., 11., 25..

PO: 1., 3., 7., 9., 24..

As a global result these formulations appear to be distributed in the following way on the windrose graphic representing the four MSTs.



The unequal distribution over the four quadrants has to be stressed. However, it is a remarkable fact that the alternative quadrant to that of the dominant formulation is not void and rather it includes, as one could expect, Heisenberg' matrix mechanics.

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