Photon-electron scattering: some contributions by Ettore Majorana

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Abstract: Relevant contributions by Majorana regarding Compton scattering off free or bound electrons are here presented, where a full quantum generalization of the Kramers-Heisenberg dispersion formula is derived. The role of intermediate electronic states is pointed out in recovering the standard Klein-Nishina formula (for free electron scattering) by making recourse to a limpid physical scheme. For bound electron scattering, a quantitative description of the broadening of the Compton line is obtained for the first time by introducing a finite mean life for the excited state of the electron system. Finally, a generalization aimed to describe Compton scattering assisted by a non-vanishing applied magnetic field is considered.

Keywords: Ettore Majorana, free electron scattering, bound electron scattering, magnetic field.

1. Introduction

Among the different phenomena that paved the way to the emergence of the quantum world, the Compton effect certainly played a key role in the acceptance of the photon as the quantum counterpart of an electromagnetic wave (Compton 1961). Indeed, a standard kinematical analysis of this process, just based on the relativistic energy-momentum conservation, directly led to the Compton formula for the wavelength shift. However, already Compton realized in his experiments that the appearance of an incoherent scattered radiation with a different frequency, in addition to the coherent scattering radiation with the same frequency, was not the only novelty with respect to the classical Thomson scattering of soft X-rays. A key result was the increasing of the relative importance of the incoherent scattering with the hardness of the X-rays employed and with the corresponding scattering angle: in particular, for very hard X-rays impinging on an atomic substance with small atomic weight, at large scattering angles practically only the incoherent radiation is present.

The intensity of the scattering of light waves by a charged particle with mass *m* and charge *e* was earlier calculated within Maxwell electrodynamics by Thomson in 1904, obtaining the classical value for the total cross-section:

$$\sigma = \frac{8\pi}{3}\frac{e^4}{m^2c^4} . \tag{1}$$

However, scattering of hard radiation did not follow this simple relation, and Compton himself in 1923 proposed an *ad hoc* formula (Compton 1923) within a classical picture, but with some non-classical ingredients about the frequency shift in order to take into account the experimental observations. Some pioneering works then followed, by Kramers and Heisenberg in 1925 (Kramers, Heisenberg 1925), who succeeded in obtaining from the correspondence principle a dispersion formula for the radiation scattered by atoms, and by Dirac (1926) and, independently, Gordon (1926). For the first time the latter authors applied quantum mechanics to the Compton problem: the quantized current of the (scalar) electron was calculated (by means of the Schrödinger or Klein-Gordon equation, respectively) and then used as source of the retarded potential entering the classical expression for the scattering intensity.

Klein and Nishina (1929) shared the same strategy of Gordon, but the appearance at the start of 1928 of the Dirac equation allowed the electrons to be described by this relativistic equation, with the obvious consequence that now the electron spin was automatically taken into account. In terms of the differential cross-section, i.e. the ratio of the number of scattered photons into the unit solid angle over the number of incident photons, the result was the following:

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\frac{e^4}{m^2c^4}\left(\frac{\nu'}{\nu}\right)^2\left(\frac{\nu'}{\nu}+\frac{\nu}{\nu'}-\sin^2\theta\right), \tag{2}$$

in agreement with the experimental data about the absorption of X-rays by matter but still obtained by means of a semi-classical method. A full quantum approach (quantized radiation and matter fields) appeared soon after, in 1930, when Waller (1930) and, especially, Tamm (1930) re-derived the Klein-Nishina formula in a fully consistent approach, by adopting the newly discovered quantum field theory formalism of Heisenberg and Pauli (1929). The main point was that, in contrast with the approach of Klein and Nishina, the Compton scattering revealed to be a second-order effect, where electron intermediate states are present to bridge from the photon absorption process to that of re-emission of another photon by the electron. This result followed from the application of the time-dependent perturbation scheme of Dirac (1927), the intermediate states being required by the interaction term linear in the electromagnetic field that prevents a direct transition from the initial to the final state. The surprising feature was the necessity to sum also over negative energy intermediate states of the electron in order to obtain the correct Klein-Nishina formula. Summing up, the calculation of the Compton cross-section served as a powerful illustration of the attainment of a consistent and also manifestly covariant perturbation evaluation of QED processes. Furthermore, the intensive experimental study, carried out for more than a decade following Compton's initial discovery in 1923, just supported this new theory (and disclosed novel processes, such as pair creation and annihilation), but the natural improvement and refinement of the experimental setups also led to the observation of precision effects (in those years, the name Compton effect referred to the scattering of a photon on a free electron as well as on bound electrons), which as well called for a theoretical explanation.

In the present contribution we unveil the unknown contributions about this subject made by Ettore Majorana around the same years (end of 1920s), as resulting from the study of his unpublished research notes (Esposito *et al.* 2003; Esposito *et al.* 2008). The interest in them, indeed, is not only historical in nature but, as common for this author, pertains also to modern theoretical physics research, given the particularity of the approaches employed, both for the time and for today.

2. Free electron scattering

The key idea of the scattering process as a series of successive absorption and emission processes, introduced by Waller and Tamm, is at the basis also of the computations performed by Majorana (Esposito *et al*. 2008, p. 104). It is not known whether he was aware of the papers of those authors, with whom he shared also the general application of Dirac's theory of dispersion to the radiation scattering problem, but, as a matter of fact, Majorana's approach to the calculation is quite different from Waller's and Tamm's.

As in Waller, the interaction between the quantized electromagnetic radiation and free electrons is described by the Dirac equation, which is, then, the starting point also of Majorana's calculations. In order to perform perturbative calculations, the complete Hamiltonian of the system considered by Majorana is split in two parts as follows:

$$\begin{aligned} H &= H_0 + I \\ H_0 &= -c\rho_1 \boldsymbol{\sigma}\cdot\mathbf{p} - \rho_3 mc^2 + \sum_s \hat{n}_s h\nu_s \quad , \\ I &= -e\rho_1 \boldsymbol{\sigma}\cdot\mathbf{C} \end{aligned} \tag{3}$$

where $\boldsymbol{\sigma}$ are Pauli matrices, ρ_1 and ρ_3 are suitable 4×4 block matrices characterized by the presence of the 2-dimensional identity matrix on each off diagonal block and diagonal block, respectively, $\mathbf{C}$ is the electromagnetic field operator and $\hat{n}_s$ is the number operator. Let us notice that, in the free term H_0, a quantized electromagnetic field contribution has already been taken into account.

Since the Compton process is the scattering of one photon off one electron, the initial and final states in Majorana's calculations are given by $|1\rangle = |a;1,0\rangle$ and $|2\rangle = |b;0,1\rangle$, respectively, where a and b label the initial and final electron states, while the photon occupation numbers refer just to the generic s-th and t-th modes of the quantized radiation field, the other modes being empty. Then he found that the matrix element of the perturbation Hamiltonian I between these two states vanishes, so that no first-order contribution is present, as realized also by Waller and Tamm. The necessity to push the approximation up to second-order terms evidently call for the presence of intermediate states in the matrix element calculations, but here Majorana – differently from Waller and Tamm – realized that only two possible intermediate states exist that lead to non-vanishing matrix elements of the perturbation Hamiltonian. They refer to 0- and 2-photon states. Let us recall that, in the modern QED formalism, the existence of only two (at second-order) intermediate states corresponds to the fact that only two Feynman diagrams contribute to the Compton scattering. Majorana then proceeded with his calculations by applying Dirac's time-dependent perturbation theory, just as Waller and Tamm did, and obtained the following dispersion formula for the transition probability from the initial state to the final one:

$$P_{12} = |a_2|^2 = 4\frac{\sin^2\left[\pi(E_2 - E_1)t/h\right]}{(E_2 - E_1)^2}\left|\sum_k \frac{I_{2k}I_{k1}}{E_k - E_1}\right|^2 , \tag{4}$$

where the summation is over all the intermediate states. The pre-factor in the above formula is sharply peaked around E_2 - E_1 = 0, so that the dominant contribution to the probability comes from near the resonance $E_1 \sim E_2 = 0$, obviously ensuring the conservation of energy. The subsequent calculations of the transition probability for the Compton process make explicit reference to both positive and negative energy states for the intermediate states and Majorana's final result (averaged over initial and summed over final electron spins and photon polarizations) reproduces the standard Klein-Nishina formula

$$P_{12} \propto \left(\frac{\nu_t}{\nu_s} + \frac{\nu_s}{\nu_t} - \sin^2\theta\right). \tag{5}$$

3. Bound electron scattering

In the decade following the Compton discovery it became clear that X-ray photons interact with atomic electrons substantially in three different ways: 1) the photon may be coherently scattered and no change intervenes in the electron state; 2) the photon may be incoherently scattered by the electron, which undergoes a transition to a continuum state; 3) the photon may be incoherently scattered by the electron, which jumps to another bound state. As a consequence, for softer radiation the electron binding had to be taken into account and a relaxation of the basic assumptions behind the Klein-Nishina formula was required. Furthermore, it was detected that the probability of incoherent scattering is decreased at low scattering angles: the energy transfer to a bound atomic electron is suppressed unless the electron gains that amount of energy required for a transition to some available higher energy state. Conversely, the probability of coherent scattering is increased at low angles since, for increasing binding energy, the whole atom absorbs photon momentum, and the probability for coherent Rayleigh scattering increases. It was realized that, for extremely large binding, the Thomson scattering cross section is recovered.

The theoretical analysis of such effects started very early, but remained at a semi-quantitative stage, lacking a general Klein-Nishina like formula for scattering off bound electrons, which of course depends definitively on the given binding, i.e. on the particular atom considered. In 1927, by the use of non-relativistic quantum mechanics, Wentzel (1927) succeeded in obtaining a generalization of the Kramers-Heisenberg dispersion formula to low-energy X-rays and bound electrons (incoherent scattering), showing that the modified line for bound electron scattering is a small continuous spectral distribution ascribed to scattering electrons whose initial state is a discrete (negative energy) level and whose final state is one of the continuum (positive energy) levels. Wentzel's dispersion formula was corrected, for some peculiarities of incoherent radiation, two years later by Waller and Hartree (1929), who performed a quantum mechanical calculation of the intensity of total (coherent plus incoherent) scattering of X-rays by atoms of a mono-atomic gas. The result was that the many-electron problem cannot be obtained as the sum of one-electron problems, since several transitions are forbidden by Pauli principle, some final states being not allowed by it. In the same influential paper of 1930 (Waller 1930) providing the first quantum derivation of the Klein-Nishina formula, Waller considered as well the scattering off bound electrons, but neglecting relativistic and spin effects, and without going further in particular calculations. The major general achievement came in 1934, when Bloch (1934) relaxed Wentzel's original assumptions for bound electrons, by describing the motion of the electrons in the atom by hydrogen wave-functions.

Majorana did not address all the open questions mentioned above (Esposito *et al.* 2008, p. 112) but, quite interestingly, he provided quantitative general results in particular cases, whose physical interpretation has revealed to be long lasting and particularly illuminating. In particular he dealt with the scattering of photons on a system of f bound electrons described by the collective wave-function $\psi_a(\mathbf{q}_1,\ldots,\mathbf{q}_f)$ with energy E_a (a labels the corresponding state of the electronic system). He started with the computation of the matrix elements of the interaction Hamiltonian (see Eq. (3), third line) between states whose number of photons in a given mode s differs by one and adopted the long wavelength approximation for the incident radiation. The physical situation he considered was that with the same initial and final energy of the electron system; that is, if the system goes into an excited state, it re-emits exactly the excitation energy. Thus, two possible ways exist for the process to occur, corresponding to two different intermediate states: 1) resonant scattering case, with the intermediate state containing no photons; 2) non-resonant scattering case, with the intermediate state containing two photons.

Then, again by means of standard time-dependent perturbation theory, he got the following formula for the transition probability between the initial and the final state:

$$P_{12} = |a_2|^2 = 4\frac{\sin^2\left[\pi\left(E_2 - E_1\right)t/h\right]}{\left(E_2 - E_1\right)^2}\left|\sum_k \frac{I_{2k} I_{k1}}{E_k - E_1 + (h/4\pi i T)}\right|^2, \qquad (6)$$

which generalizes Eq. (4) to the case of a non vanishing lifetime T. Also here the dominant contribution to the probability comes from near the resonance $E_1 \sim E_2 = 0$.

For some unknown reason, Majorana went on in his analysis by including the effect of a time-varying magnetic field on the bound electrons system. Such a problem was not in the agenda (at the time) for those experimenting on Compton scattering and, then, also for theoretical physicists, so that it is unlikely that Majorana was here stimulated by practical problems. Indeed, any appreciable influence of a magnetic field on the Compton process has some chance to manifest only for very strong magnetic fields, such as – in the laboratory case, for sinusoidal fields – for laser-assisted scattering[1] or, rather, in astrophysical environments (Gonthier *et al.* 2000). Even for the simplest case of a constant magnetic field, the only indirect effect is through polarization effects on the electron system (Franz 1938) – that is the magnetic field interacts directly with the electrons, upon which the Compton scattering takes place – but, again, such phenomenon was considered only later and, in any case, was not the main concern of Majorana's calculations. Instead, Majorana's work seems to have some contact points with another paper of him (Majorana 1932), published some years later and related to a different experimental situation studied by his friend and colleague Segrè (Recami 2011; Esposito 2008; Esposito 2009), or even, alternatively, related to the Raman scattering studied, at the end of 1920's, by Amaldi, Segrè and Rasetti (Amaldi 1929).

4. Conclusions

At the emergence of the quantum description of Nature, quite a relevant role – though not unique – was played by the Compton process for the direct detection of photons, i.e. the quanta of the electromagnetic field, as well as for the dynamical description of the effect, which called for a suitable theoretical prediction for the scattering cross section. The experimental validation of the Klein-Nishina formula revealed that quantum mechanics applied successfully also to this electron-photon scattering problem, but the theoretical problem remained of a fully quantum description of the phenomenon, whose solution was the first bench test of the quantum field theory applied to electrodynamics. Indeed, although the QED results obtained by Waller and Tamm just confirmed the semi-classical prediction of the Klein-Nishina formula, the change of perspective was not at all negligible: Compton scattering resulted to be mediated by electronic intermediate states relating the initial photon absorption process to the final re-emission of another photon by the intervening electron. Also, for the first time, the relevant role of negative energy Dirac states was proved in order to obtain physically meaningful results. Further precision effects revealed by experiments, including scattering off bound (rather than free) electrons, also called for a thorough theoretical description but, here, detailed quantitative predictions were generally obscured by mathematical (and physical) technicalities. In this scenario, quite intriguing emerge the unpublished contributions by Majorana, whose elegant and modern treatment of the basic scattering process reveals very clearly the key physical ideas behind the phenomenon under study. Indeed, if his derivation of the Klein-Nishina formula mathematically resulted just from a full quantum form of the Kramers-Heisenberg dispersion formula he derived, the limpid physical scheme he realized mimicked quite close the later Feynman diagrams approach, based on the existence of only two intermediate electronic states at leading approximation. Major achievements were obtained also for the problem of bound electron scattering, where Majorana was able (for the first time) to quantitatively describe the broadening of the Compton line, by introducing a finite mean life of the excited state of

[1] For a recent review see (Seipt, Kampfer 2014) and references therein.

the electron system. He was probably led to such an approach by his own pioneering studies about quasi-stationary states in nuclear physics. Finally, and even more intriguing, Majorana unexpectedly studied also the scattering process by bound electrons when a non-vanishing time-varying magnetic field is applied to the system, this phenomenon being considered only in more recent times, when its relevance in some astrophysical environments and in laboratory laser-assisted scattering came out.

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