Elements of geometrical calculation in Archimedes. The laws of statics

Angelo Pagano[1,3] – angelo.pagano@ct.infn.it; Giuseppe Boscarino[3] – gpp.bos@libero.it; Oreste Caniglia[3,4] – profocaniglia@yahoo.it; Pietro Di Mauro† [1,2,3]; Emanuele V. Pagano[3,5] – epagano@lns.infn.it

[1] INFN - Sez. di Catania,
[2] Liceo Scientifico Statale "E. Fermi" – Paternò (CT)
[3] Associazione Culturale "Salvatore Notarrigo" – La Scuola Italica
[4] I.I.S. "Concetto Marchesi" di Mascalucia (CT)
[5] INFN - Laboratori Nazionali del Sud
† deceased - In Memoriam

Abstract: The work of Archimedes (Siracusa) strongly influenced the development of modern science. In this report, the laws of static are discussed within the modern "Geometrical calculus", such as the one provided by the mathematician Giuseppe Peano (Torino). The translation of Archimedes' work by symbolic logic is made possible by the unambiguous terms used by Archimedes in his postulates and propositions. The formal results are also applicable to the theory of the collisions between two impenetrable bodies, as the one described in d'Alembert's mechanics. This modern axiomatic interpretation is a clear indication of the existence in Archimedes' work of a complete set of logical rules (precursor) whose real interpretation is consistently illustrated by different physical models.

Keywords: Archimedes, Statics, d'Alembert, impenetrable body.

1. Introduction

Discussing about Archimedes in a modern physics context is a unique opportunity to remind us the centrality of Archimedes' position in human sciences (particularly in Physics). Nobel laureate Alferov illustrates very well the concept:

> Archimedes is considered one of the greatest scientists of every time. He has discovered important mathematical and physical principles, designing ingenious instruments and machines that the people of Syracuse have used to defend their homeland by Romans' attacks (growing up domination at that time). […] Archimedes was admired by distinguished ancient Philosophers (Eratosthenes, Aristarchus, Plutarch, Polybius, […]). Archimedes' thought continues to inspire great interest still nowadays and has significantly influenced the Galilean scientific work during its evolution from the original instances, based on medieval roots, to the scientific revolution of the renaissance era (Alferov 2008, pp. 12-13).

In a more technical aspects, philosopher M. Serres carefully explains the philosophical essence:

> What is detached from Aristotle is, once again, the Archimedean world: sloping tiles, static, hydraulic, differential pre-calculus. It is in the Arenario that the world is Helios - centered, with the support of Aristarchus. […] Certainly, Leonardo, Galilei, Torricelli, […] Descartes and others cut the bridges with the Middle Ages and the School (of Aristotle); but we have to recognize that Epicurus (Democritus) and Archimedes are already a non-Aristotelian universe. Modern science did not arise, suddenly, from the nothingness or the solicitations of contemporaries, during the renaissance, […] it simply was born again, that's all! It will take a long time to reach Archimedes' perfection. [...] But, the founders of modern science have learned their profession in Archimedes' work (Serres 2000).

In this paper we briefly discuss about Archimedes' important work: *On the equilibrium of planes or centres of gravity of plane figure* (book I), quoted in the following as ARCH. It is in close connection with the "geometry of masses", as the one developed by the mathematician G. Peano (1888). In this paper the book I is described by using Peano's symbolic logic.

Let us define shortly what is a *geometrical calculus* in the frame of this paper. It is recognised by four main items:

a) elements (entities) of the calculus are from Euclidean geometry (point, line, surface, volume) and from Newtonian dynamics (mass, applied forces, etc.);
b) the calculus is performed directly on abstract entities (such it is an idealized body). Co-ordinate system of reference is unnecessary;
c) "time" is an external event-ordering parameter, i.e., absolute time is assumed;
d) a system of logical rules are used to combine elements in postulates and propositions (theorems).

An important example of geometric calculus was given by G. Peano in Torino (1888) and it was extended by scholars to multi-dimensional (>3) Euclidean spaces (see for example Boggio, Burali-Forti 1924).

For a careful discussion about Archimedes' work, the reader can consult the book of Dijksterhuis (1956). From this book we read:

> The treatise on the equilibrium of planes occupies a place apart in the work of Archimedes. In fact, whereas in all his mathematical treatises he builds on foundations long ago established, in this work he concerns himself with an investigation into the very foundations; moreover he leaves the domain of pure mathematics for that of natural science considered from the mathematical point of view: he sets forth certain postulates on which he bases a chapter from the theory of equilibrium, and he is thus the first to establish the close interrelation between mathematics and mechanics,

> which was to become of such far-reaching significance for physics as well as mathematics (Dijksterhuis 1956, p. 286).

Dijksterhuis did not use logical symbols. Evidently, the main advantage to translate Archimedes' propositions in symbols of logical mathematics is given by the theoretical possibility to adapt the logical translation to different physical descriptions, obeying to the same rules. One of these examples is the theory of collisions between two impenetrable bodies as illustrated in d'Alembert's famous *Traité de dynamique* (1743).

2. Logical translation of Archimedes' static laws

In this chapter we will discuss few postulates and propositions included in Archimedes' work. The purpose is to show the potentiality of the method; a more complete discussion will be given in a separate publication. This work was inspired by the first logical translation (to our knowledge) made by S. Notarrigo (1992). However, Notarrigo did not use Peano's logical symbols. Let us to introduce some among Peano's early operators that are used in our translation:

1.	operator: IF…. THEN = (it follows) implication	$\supset$
2.	operator: direct and inverse implication	$\supseteq$
3.	operator: ET	$\wedge$
4.	operator: EQUILIBRIUM	$E_{=}$
5.	operator: NON EQUILIBRIUM*:* *INCLINE towards side >, or side <*	$E_{>}; E_{<}$
6.	operator: X is one of the elements of class A	$X \in A$
7.	Measurable Quantity	Q
8.	Real number	R
9.	Operator: for all	$\forall$
10.	Multiplier real number and quantity	*
11.	Parenthesis	(…)
12.	Separators between propositions	$\therefore$
13.	Identity	$\equiv$
14.	Ratio	/

Beside, we adopt two basic properties (**BP.i**, i=1,2), as due to G. Peano (1908), and (implicitly) used in Archimedes' work:

BP.1 $Q \equiv quantity$
$\forall q \in Q \therefore \forall x \in R \supset x * q \in Q$

BP.2 equality fundamental properties: symmetry, reflection and transitivity:
$x = x \therefore x = y \supset y = x \therefore (x = y) \wedge (y = z) \supset x = z$

In ARCH, seven postulates (**Pi**, i=1,7) are given in agreement with Dijksterhuis' classical work. In this paper we discuss four postulates **P.I, P.II, P.III, P.VI**. They concern with discrete (masses) bodies.

Postulates:

I- Equal weights (p_i , i=1,2) (suspended) at equal distances (d_i , i=1,2) are in equilibrium, and equal weight at unequal distances are not in equilibrium, but incline toward the weight which is at the greater distance.

Logical translation:

$$\left(\frac{p_1}{p_2}=1\right)\supset\left\{\left[\left(\frac{d_1}{d_2}=1\right)\supseteq E_{=}\right]\wedge\left[\left(\frac{d_1}{d_2}>1\right)\supseteq E_{>}\right]\wedge\left[\left(\frac{d_1}{d_2}<1\right)\supseteq E_{<}\right]\right\} \quad \textbf{P. I}$$

II- that if, when weights at certain distances are in equilibrium, something be added to one of the weights, they are not in equilibrium, but incline towards that weight to which something has been added

III- Similarly that, if anything be taken away from one of the weights, they are not in equilibrium, but incline towards that weight from which nothing has been taken away.

VI- If magnitudes at certain distances be in equilibrium, (magnitudes) equal to them will also be in equilibrium at the same distances.

Logical translation:

$$\left[\left(\frac{p_1}{p_2}=a\right)\wedge\left(\frac{d_1}{d_2}=\alpha\right)\supset E_{=}\right]\supset$$

$$\left[\left(\frac{p_1}{p_2}>a\right)\wedge\left(\frac{d_1}{d_2}=\alpha\right)\supset E_{>}\right]\wedge \quad \textbf{P. II}$$

$$\left[\left(\frac{p_1}{p_2}<a\right)\wedge\left(\frac{d_1}{d_2}=\alpha\right)\supset E_{<}\right]\wedge \quad \textbf{P.III}$$

$$\left[\left(\frac{q_1}{q_2}=a\right)\wedge\left(\frac{d_1}{d_2}=\alpha\right)\supset E_{=}\right] \quad \textbf{P.VI}$$

Once the formal structure is established, the "real ≡ physics" interpretation is ‘’ external to the logical system" and it should be taken by phenomena. It follows that formulation can be adapted to different phenomena. We will see in the following chapter a relevant application, i.e., d'Alembert's early theory of collisions. Starting from postulates **P.I-P.VI** and using basic logical rules it is possible to proof different propositions (theorems) about statics. We proof just one of them in the following.

Proposition (theorem) I:

Weights which are in equilibrium at equal distances are equal.

Logical translation:

$$\left[\left(\frac{p_1}{p_2} = a\right) \wedge \left(\frac{d_1}{d_2} = 1\right) \wedge E_{=}\right] \supset \left[\left(\frac{p_1}{p_2} = a = 1\right)\right] \qquad \textbf{Th. 1}$$

Proof (*reductio ad absurdum*):

For, if they were unequal (it means: in contrast with the Th.1), by taking away from the greater weight a quantity by which it exceeds the lesser, we disturb the equilibrium on account of postulate III, whereas because of postulate I there would precisely have to be in equilibrium in the new position.

Logical translation:

let be (by absurd) **a>1** , it follows:

$$\left[\left(\frac{p_1}{p_2} = (a > 1)\right) \wedge \left(\frac{d_1}{d_2} = 1\right) \supset E_{=} \;\; (hypotesis\ in\ Th.1\ and\ a > 1)\right]$$

(as required by **P.III**) $$\supset \left[\left(\frac{p_1}{p_2} = (1 < a)\right) \wedge \left(\frac{d_1}{d_2} = 1\right) \supset E_{<}\right]$$

This (right hand) result is not in agreement with the postulate **P.I**; consequently, **Th.1** is proved. C.V.D

Archimedes' laws, culminate with the (so called) "lever principle" (not discussed in detail in this paper):

Th. VI : Commensurable magnitudes are in equilibrium at distances reciprocally (inverse) proportional to the weights

Th. VII: However, even if magnitudes are incommensurable, they will be in equilibrium at distances reciprocally (inverse) proportional to the magnitudes.

As commonly stated in textbooks of elementary physics:

$$\boldsymbol{p_1 / p_2 = d_2 / d_1}$$

or

$$\boldsymbol{p_1 * d_1 = p_2 * d_2}$$

3. d'Alembert's early Collision-Theory

The theory of impenetrable (solid bodies) has been described by d'Alembert in his masterpiece *Traité de Dynamique*. This theory is not more in use in classical physics because the concepts of "impenetrability" and "solid bodies" have been abandoned in modern paradigm. Modern paradigm is based on the notions of “point-mass particles’’ and “rigid bodies”. The reasons of this change are quite complex and, basically, they are linked with the abandon, in modern paradigm, d’Alembert’s *ab-initio* definition of body (mass):

> Si deux portions d’étendue semblables & égales entr’elles sont *impénétrables*, c’est-à-dire, si elles ne peuvent être imagines unies & confondues l’une avec l’autre, de manière qu’elles ne fassent qu’une même portion d’étendue moindre que la somme des deux, chacune de ces portions d’étendue sera ce qu’on appelle un *Corps*. L’impénétrabilité est la propriété principale par laquelle nous distinguons les Corps des parties de l’espace indéfini, où nous imaginons qu’ils sont placés (d’Alembert 1743, p. 1).

We simple notice that this latter definition of mass is in full agreement with Newton’s theoretical concepts, as the product between the density and volume (Pagano 2011). By changing the physical interpretation of elements in Archimedes’ postulates, we are able to account for d’Alembert’s collisions theory. As an example, changing: “weight” (p) with “mass” (m) and “distance” (d) with “velocity” (v), consequently, the postulate **P.I** is written in the following:

$$\left(\frac{m_1}{m_2}=1\right)\supset\left\{\left[\left(\frac{v_1}{v_2}=1\right)\supseteq E_=\right]\wedge\left[\left(\frac{v_1}{v_2}>1\right)\supseteq E_>\right]\wedge\left[\left(\frac{v_1}{v_2}<1\right)\supseteq E_<\right]\right\} \quad \textbf{P'.I}$$

And we read:

P’.I Equal masses (m_i , i=1,2) moving (one against the other) with equal velocities (v_i , i=1,2) are in equilibrium, and equal masses moving with unequal velocities are not in equilibrium, but they move toward the mass which has the larger (between the two ones) velocity.

$$\left[\left(\frac{m_1}{m_2}=a\right)\wedge\left(\frac{v_1}{v_2}=\alpha\right)\supset E_=\right]\supset$$

$$\left[\left(\frac{m_1}{m_2}>a\right)\wedge\left(\frac{v_1}{v_2}=\alpha\right)\supset E_>\right]\wedge \quad \textbf{P'. II}$$

$$\left[\left(\frac{m_1}{m_2}<a\right)\wedge\left(\frac{v_1}{v_2}=\alpha\right)\supset E_<\right]\wedge \quad \textbf{P'. III}$$

$$\left[\left(\frac{q_1}{q_2}=a\right)\wedge\left(\frac{v_1}{v_2}=\alpha\right)\supset E_=\right] \quad \textbf{P'. VI}$$

we read:

Let be masses in equilibrium moving (one against the other) with certain velocities; it follows:

P’.II - Something is added to one of the masses: equilibrium is broken. Masses move towards that mass-direction to which nothing has been added.

P’.III- Similarly that, if something is taken away from one of the masses: equilibrium is broken. Masses move towards that mass-direction from which something has been taken away.

P’.VI- If magnitudes with certain velocities are in equilibrium, other magnitudes (equal to them) are also in equilibrium with the same velocities.

The semantic expression: “equilibrium” is simply interpreted by the one: “common centre of mass is at rest”; consequently, postulates **P’.I, P’.II, P’.III** and **P’.VI** describe the (head-on) collisions (of d’Alembert) between two impenetrable bodies. Propositions concerning collisions are deduced (in perfect correspondence with the above mentioned laws of static). In particular, **Th. VI** and **Th. VII** of the previous chapter concerning “weights” in “equilibrium” are applied to the collisions of two bodies, and both are summarised in one single proposition (Centre of mass condition for equilibrium):

$$\boldsymbol{m_1 / m_2 = v_2 / v_1}$$

or

$$\boldsymbol{m_1 * v_1 = m_2 * v_2}$$

4. Conclusions

The Archimedean laws of statics have been interpreted in the context of a theory of *geometrical calculus*, as the one developed by G. Peano (XIX century) and, consequently, translated in symbols of logical mathematics. In this form they reveal their validity as coherent system of axioms and theorems that find applications in different fields. In particular in this paper, the correspondence between the Archimedean laws of statics and d’Alembert’s early collisions theory is argued. This modern axiomatic interpretation is a clear indication of the existence in Archimedes’ work of a complete set of logical rules (precursor) whose real interpretation is, consistently, illustrated by different physical models.

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